Solution Sheet 6

Exercise 6.1

Let $0 < \alpha < \beta \leq 1$. Prove that $C^{0,\beta}$ compactly embeds into $C^{0,\alpha}$.

Proof. Letting (f_n) be a bounded sequence in C^β, we must show that there exists a subsequence (f_{n_k}) which is convergent in C^α. We identify the subsequence via the Arzelá-Ascoli Theorem, as each f_n is uniformly continuous, we can continuously extend to the closure of its domain such that the extended function is again uniformly continuous. Deducing pointwise uniform boundedness and equicontinuity out of the uniform boundedness in C^β, we can apply Arzelá-Ascoli to deduce the existence of a uniformly convergent subsequence (f_{n_k}). We demonstrate convergence in C^α by showing that the sequence is Cauchy; we employ the inequality shown in the lecture notes,

$$\|f_{n_k} - f_{n_j}\|_\alpha \leq \|f_{n_k} - f_{n_j}\|_\beta^{\frac{\alpha}{\beta}} \left(2\|f_{n_k} - f_{n_j}\|_\infty\right)^{1-\frac{\alpha}{\beta}} \leq C\|f_{n_k} - f_{n_j}\|_\infty^{1-\frac{\alpha}{\beta}}$$

using in the second inequality the uniform boundedness in C^β. The Cauchy property thus follows from the Cauchy property in supremum norm, concluding the result.

□

Exercise 6.2

Let $0 < \alpha < 1$. Give an example of a function f such that $f \in C^\alpha$ however $f \notin C^\beta$ for all $\alpha < \beta < 1$.

Proof. Our example is $f : (0,1) \to \mathbb{R}$, $f(x) = x^\alpha$. Then $f \in C^\alpha$ as

$$\frac{|f(u) - f(v)|}{|u - v|^\alpha} = \frac{|u^\alpha - v^\alpha|}{|u - v|^\alpha} \leq 1.$$

To prove that $f \notin C^\beta$ take $u = v + \varepsilon$, then

$$\frac{|f(u) - f(v)|}{|u - v|^\beta} = \frac{(v + \varepsilon)^\alpha - v^\alpha}{\varepsilon^\beta} \geq \varepsilon^{\alpha-\beta}$$

which explodes as $\varepsilon \to 0$. □

Exercise 6.3

Suppose that f is uniformly continuous. Does there exist an $\alpha > 0$ such that $f \in C^\alpha$?

Proof. Yes, consider $f : [0, \frac{1}{2}] \to \mathbb{R}$, $f(x) = \frac{1}{\log(x)}$ if $x > 0$ and $f(0) = 0$, which is continuous hence uniformly continuous. However, it cannot belong to C^α: considering $v = 0$,

$$\frac{\log(u)}{u^\alpha}$$

which explodes as $u \to 0$. □

Exercise 6.4

Let W be a standard real-valued Brownian Motion on $[0,1]$, and $\alpha > \frac{1}{2}$. Prove that for $\mathbb{P} - a.e.$ ω, $W(\omega) \notin C^\alpha$.

Proof. We use that W has two-variation equal to t, or more precisely that for partitions $\Delta^n : 0 = t_0^n < t_1^n < \cdots < t_{M_{n+1}}^n = t$ of $[0,t]$ with mesh $|\Delta^n| \to 0$, then for $\mathbb{P} - a.e.\ \omega$, the sequence $(T_n(\omega))$ defined by

$$T_n(\omega) = \sum_{i=0}^{M_n} |W_{t_{i+1}^n}(\omega) - W_{t_i^n}(\omega)|^2$$

has a convergent subsequence to t. We now assume for a contradiction that $W(\omega) \in C^\alpha$ for $\alpha > \frac{1}{2}$, in particular $\left|W_{t_{i+1}^n}(\omega) - W_{t_i^n}(\omega)\right| \leq C(\omega)|t_{i+1}^n - t_i^n|^\alpha$. Then

$$\begin{aligned}
\sum_{i=0}^{M_n} |W_{t_{i+1}^n}(\omega) - W_{t_i^n}(\omega)|^2 &\leq C^2(\omega) \sum_{i=0}^{M_n} |t_{i+1}^n - t_i^n|^{2\alpha} \\
&\leq C^2(\omega)|\Delta^n|^{2\alpha-1} \sum_{i=0}^{M_n} (t_{i+1}^n - t_i^n) \\
&\leq C^2(\omega) t |\Delta^n|^{2\alpha-1} \\
&\to 0
\end{aligned}$$

as $2\alpha - 1 > 0$, contradicting the convergent subsequence to t.

□

Exercise 6.5

Let B^H be a Fractional Brownian Motion on $[0,T]$ with Hurst exponent $H \in (0,1]$. Prove that for a given $\alpha < H$, there exists a modification $\tilde{B}^M$ of B^M such that for $\mathbb{P} - a.e.\ \omega$, $\tilde{B}^M(\omega) \in C^\alpha$.

Proof. The idea is, unsurprisingly, to apply the Kolmogorov Continuity Theorem. Due to the stationary increments and $H-$self-similarity of B^H,

$$\mathbb{E}\left(|B_t^H - B_s^H|^p\right) \leq \mathbb{E}\left(|B_1^H|^p\right)|t-s|^{pH}$$

and in the notation of the Kolmogorov Continuity Theorem,

$$\|B_t^H - B_s^H\|_p \leq \|B_1^H\|_p |t-s|^H.$$

Thus for p sufficiently large $H > \frac{1}{p}$, hence B^H admits a modification in C^γ for every $\gamma < H - \frac{1}{p}$, and in particular in C^α for p large enough such that $\alpha < H - \frac{1}{p}$.

□

Exercise 6.6

Prove that a real-valued stochastic process X with independent increments is a Markov Process.

Proof. It is sufficient to show that, for every bounded measurable $f : \mathbb{R} \to \mathbb{R}$,

$$\mathbb{E}\left(f(X_t)|\mathcal{F}_s\right) = \mathbb{E}\left(f(X_t)|X_s\right)$$

(in fact, this is equivalent to the Markov Property). We rewrite

$$f(X_t) = f(X_t - X_s + X_s) = g(X_t - X_s, X_s)$$

where $g(u,v) = f(u+v)$. Thus

$$\mathbb{E}\left(f(X_t)|\mathcal{F}_s\right) = \mathbb{E}\left(g(X_t - X_s, X_s)|\mathcal{F}_s\right) = \mathbb{E}\left(g(X_t - X_s, X_s)|X_s\right) = \mathbb{E}\left(f(X_t)|X_s\right)$$

having used that $X_t - X_s$ is independent of $\mathcal{F}_s$, and that X_s is measurable with respect to $\mathcal{F}_s$, in the middle step. □

Exercise 6.7

Let (e_i) be an orthonormal system of a separable Hilbert Space H, (λ_i) a sequence of summable positive constants, and $(B^{H,i})$ a sequence of real-valued i.i.d, fractional Brownian Motions with Hurst exponent H on $[0,T]$. Define, for each $n \in \mathbb{N}$ and $t \in [0,T]$,

$$X_t^n = \sum_{i=1}^n \sqrt{\lambda_i} e_i B_t^{H,i}.$$

1. Show that for each $t \in [0,T]$, (X_t^n) converges to some (X_t) in $L^2(\Omega; H)$.
2. Prove that the process X defined in the previous part is Gaussian.
3. Prove that for a given $\alpha < H$, there exists a modification $\tilde{X}$ of X such that for $\mathbb{P} - a.e.\ \omega$, $\tilde{X}(\omega) \in C^\alpha\left([0,T]; H\right)$. You are given that the notion of Hölder Continuity and Kolmogorov's Continuity Theorem both extend to H, as well as the Kahane-Khintchine Inequality which states that for Y an $H-$valued Gaussian random variable, for all $1 \leq p, q < \infty$,

$$\left[\mathbb{E}\left(\|Y\|^p\right)\right]^{\frac{1}{p}} \leq c_{p,q}\left[\mathbb{E}\left(\|Y\|^q\right)\right]^{\frac{1}{q}}.$$

Proof.

1. We argue that the sequence is Cauchy, as

$$\mathbb{E}\left(\|X_t^n - X_t^m\|^2\right) = \mathbb{E}\left(\sum_{i=m+1}^n \lambda_i |B_t^{H,i}|^2\right) = \mathbb{E}\left(|B_1^H|^2\right) t^{2H} \sum_{i=m+1}^n \lambda_i$$

   which is Cauchy as the (λ_i) are summable.

2. We must prove that the finite dimensional distributions of X are Gaussian measures on H. That is to say for every $h \in H$ we must show that the finite dimensional distributions of $\langle X, h\rangle$ are Gaussian measures on $\mathbb{R}$. Fixing any collection of time points $t_1 < \cdots < t_d$, note that the $\mathbb{R}^d$-valued random variable $(\langle X_{t_1}, h\rangle, \dots, \langle X_{t_d}, h\rangle)$ is the $L^2(\Omega; \mathbb{R}^d)$ limit of $\left(\langle X_{t_1}^n, h\rangle, \dots, \langle X_{t_d}^n, h\rangle\right)$ by using Cauchy-Schwarz to see that

$$\sum_{j=1}^d \langle X_j - X_{t_j}^n, h\rangle^2 \leq \|h\|^2 \sum_{j=1}^d \|X_j - X_{t_j}^n\|^2$$

   and the previous part. Note that

$$\langle X^n, h\rangle = \sum_{i=1}^n \sqrt{\lambda_i}\langle e_i, h\rangle B^{H,i}$$

   is a Gaussian process as the sum of independent Gaussian processes, hence $(\langle X_{t_1}, h\rangle, \dots, \langle X_{t_d}, h\rangle)$ is the $L^2(\Omega; \mathbb{R}^d)$ limit of an $\mathbb{R}^d$-valued Gaussian random variable. Thus by Lemma 2.5.2, as convergence in L^2 implies convergence in distribution, $(\langle X_{t_1}, h\rangle, \dots, \langle X_{t_d}, h\rangle)$ is Gaussian as required.

3. Using the given Kahane-Khintchine Inequality for any $p \geq 1$, combined with the previous two parts,

$$\begin{aligned}
\left[\mathbb{E}\left(\|X_t - X_s\|^p\right)\right]^{\frac{1}{p}} &\leq c_{p,2}\left[\mathbb{E}\left(\|X_t - X_s\|^2\right)\right]^{\frac{1}{2}} \\
&= \lim_{n\to\infty} c_{p,2}\left[\mathbb{E}\left(\|X_t^n - X_s^n\|^2\right)\right]^{\frac{1}{2}} \\
&= \lim_{n\to\infty} c_{p,2}\left[\sum_{i=1}^{n} \lambda_i \mathbb{E}\left(|B_t^{H,i} - B_s^{H,i}|^2\right)\right]^{\frac{1}{2}} \\
&\leq \lim_{n\to\infty} c_{p,2}\left[\sum_{i=1}^{n} \lambda_i \mathbb{E}\left(|B_1^H|^2\right)|t-s|^{2H}\right]^{\frac{1}{2}} \\
&\leq \left(\lim_{n\to\infty} c_{p,2}\left[\sum_{i=1}^{n} \lambda_i \mathbb{E}\left(|B_1^H|^2\right)\right]^{\frac{1}{2}}\right)|t-s|^H \\
&= c|t-s|^H
\end{aligned}$$

The result now follows exactly as in Exercise 6.5.

□